arXiv:1504.00596v2 [math.CO] 15 Apr 2015

Extremal properties of flood-filling games

Kitty Meeks *

Dominik K. Vu †

Abstract

The problem of determining the number of "flooding operations" required to make a given coloured graph monochromatic in the one-player combinatorial game Flood-It has been studied extensively from an algorithmic point of view, but basic questions about the maximum number of moves that might be required in the worst case remain unanswered. We begin a systematic investigation of such questions, with the goal of determining, for a given graph, the maximum number of moves that may be required, taken over all possible colourings. We give two upper bounds on this quantity for arbitrary graphs, which we show to be tight for particular classes of graphs, and determine this maximum number of moves exactly when the underlying graph is a path, cycle, or a blow-up of a path or cycle.

1 Introduction

Flood-It is a one-player combinatorial game, played on a coloured graph. The goal is to make the entire graph monochromatic ("flood" the graph) with as few moves as possible, where a move involves picking a vertex v and a colour d, and giving all vertices in the same monochromatic component as v colour d. Implementations of the game, played on a square grid, are widely available online, and include a flash game [3] as well as popular smartphone apps [1, 2]. Mad Virus [4] is a version of the same game played on a hexagonal grid, and the Honey Bee Game [5] is a two player variant also played on a hexagonal grid. More generally, when played on a planar graph, the game can be regarded as modelling repeated use of the flood-fill tool in Microsoft Paint.

Questions arising from this game have received considerable attention from a complexity-theoretic perspective in recent years [6, 7, 8, 9, 10, 12, 13, 14, 15], with such work focussing on questions of the form, "Given a graph G from a specified class, and a colouring ω of the vertices of G, what is the computational complexity of determining the minimum number of moves required to flood G?" The problem is known to be NP-hard in many situations, provided that at least three colours are present in the initial colouring, including in the case that G is an $n \times n$ grid (as in the original version of the game) [6] and the case in

*University of Glasgow, kitty.meeks@glasgow.ac.uk

†University of Memphis, dominik.vu@memphis.edu

which G is a tree [8, 11] (the parameterised complexity of the problem restricted to trees has also been studied [7]). On the other hand, there are known to be polynomial-time algorithms to determine the minimum number of moves required if G is a path or a cycle, or more generally for any graph if the initial colouring uses only two colours. A more complete description of the complexity landscape for these problems can be found in [14].

Two different versions of the game have been considered in the literature, known as the "fixed" and "free" versions. In the fixed version of the game (as in most implementations), players must always change the colour of the monochromatic component containing a single distinguished vertex, so the only choice is what colour to assign to this component; in the free version players can choose freely at each move the component whose colour is changed, in addition to the new colour. For the remainder of this paper we shall only be concerned with the free version of the game.

In this paper, we initiate a systematic investigation of a different type of question about the game, which has yet to be addressed in the literature: given a graph G, what is the maximum number of moves we may need to flood G, taken over all possible colourings of G with c colours? In this paper we begin by giving two different upper bounds on the answer to this question that are valid for all connected graphs, before going on to consider various special classes of graphs and demonstrating that the upper bounds are tight for certain families of graphs. We are thus able to determine the precise answer to the question for paths, and also for cycles and blow-ups of paths and cycles. This work provides a partial answer to an open question of Meeks and Scott [13].

The rest of the paper is organised as follows. In the remainder of this section, we introduce some key notation and definitions, and mention some results from the existing literature that are of particular relevance to addressing extremal problems. In Section 2 we prove two upper-bounds on the number of moves required to flood any coloured connected graph. We continue to prove the tightness of these bounds as we consider separately trees, paths, cycles in Section 3, and prove our main result that blow-ups of paths and cycles behave like the graphs they were blown-up from, provided there are not too many colours, in Section 4.

1.1 Notation and definitions

For any graph G, we denote by $|G|$ the number of vertices in G (so $|G| = |V|$). We write $\mathcal{T}(G)$ for the set of spanning trees of G. For $u, v \in V(G)$, we let $\mathcal{P}(u, v)$ be the set of u-v paths in G, and the *distance* $d(u, v)$ from u to v is defined to be $\min_{P \in \mathcal{P}(u,v)} |P| - 1$. For any graph G, we then define the *radius* of G to be $\min_{u \in V(G)} \max_{v \in V(G)} d(u, v)$.

A graph $G = (V_G, E_G)$ is said to be a *blow-up* of a graph $H = (V_H, E_H)$ if V_G can be partitioned into sets $\{V_u : u \in V_H\}$ such that $v_1 v_2 \in E_G$ if and only if $v_1 \in V_u$ and $v_2 \in V_w$ with $uw \in E_H$.

Suppose the game is played on a graph $G = (V, E)$, equipped with an initial colouring $\omega : V \to C$ (not necessarily a proper colouring); we call C the *colour-set*. A single move $m = (v, d)$ (where $v \in V$ and $d \in C$) involves assigning colour d to all vertices in the same monochromatic component as v. Given a colouring ω with colour-set C and any $d \in C$, we denote by $N_d(G, \omega)$ the number of vertices $v \in V$ such that $\omega(v) = d$.

For any graph G with colouring ω, we can obtain a new graph and corresponding colouring by *contracting monochromatic components* of G with respect to ω, that is

repeatedly contracting an edge $e = uv$ such that $\omega(u) = \omega(v)$. If G' (with colouring ω') is obtained from G (with initial colouring ω) in this way, it is clear that any sequence of moves that floods G with initial colouring ω will also flood G' with initial colouring ω', and vice versa (up to possibly changing the vertex at which a move is played to another vertex in the same monochromatic component with respect to ω). We say that the graph G_1 with colouring ω_1 is *equivalent* to the graph G_2 with colouring ω_2 if the coloured graphs obtained by contracting monochromatic components of G_1 with respect to ω_1 and contracting monochromatic components of G_2 with respect to ω_2 are identical.

We define $m_G(G, \omega, d)$ to be the minimum number of moves required to give all vertices of G colour d, and $m_G(G, \omega)$ to be $\min_{d \in C} m_G(G, \omega, d)$. Let $\Omega(V, C)$ be the set of all surjective functions from V to C. We then define $M_c(G) = \max_{\omega \in \Omega(V, \{1,\dots,c\})} m_G(G, \omega)$.

Let A be any subset of V. We set $m_G(A, \omega, d)$ to be the minimum number of moves we must play in G (with initial colouring ω) to create a monochromatic component of colour d that contains every vertex in A, and $m_G(A, \omega) = \min_{d \in C} m_G(A, \omega, d)$. Henceforth, we omit the index should the ground graph and the area to be flooded agree, i.e. $m_G(G, \omega, d) = m(G, \omega, d)$. We say a move $m = (v, d)$ is *played in* A if $v \in A$, and that A is *linked* if it is contained in a single monochromatic component. Subsets $A, B \subseteq V$ are *adjacent* if there exists $ab \in E$ with $a \in A$ and $b \in B$. We will use the same notation when referring to (the vertex-set of) a subgraph H of G as for a subset $A \subseteq V(G)$.

1.2 Background results

One key result which we will exploit throughout this paper gives a characterisation of the number of moves required to flood a graph in terms of the number of moves required to flood its spanning trees. More precisely we will apply the following results by Meeks and Scott [14].

Theorem 1.1. *Let G be a connected graph with colouring ω from colour-set C. Then, for any $d \in C$,*

$$m(G, \omega, d) = \min_{T \in \mathcal{T}(G)} m(T, \omega, d).$$

A corollary of this result, proved in the same paper, is that the number of moves required to flood a graph H cannot be increased when moves are played in a larger graph G which contains H as a subgraph.

Corollary 1.2. *Let G be a connected graph with colouring ω from colour-set C, and H a connected subgraph of G. Then, for any $d \in C$,*

$$m_G(V(H), \omega, d) \leq m_H(H, \omega, d).$$

We will also use a simple monotonicity result for paths, proved by the same authors in a previous paper [12].

Lemma 1.3. *Let P be a path, with colouring ω from colour-set C, and let P' be a second coloured path with colouring ω', obtained from P by deleting one vertex and joining its neighbours. Then, for any $d \in C$, $m(P', \omega', d) \leq m(P, \omega, d)$. We also have $m(P', \omega') \leq m(P, \omega)$.*

Another useful result, proved in an additional paper by Meeks and Scott [13], relates the number of moves required to flood the same graph with different initial colourings.

Lemma 1.4. *Let G be a connected graph, and let ω and ω' be two colourings of the vertices of G (from colour-set C). Let $\mathcal{A}$ be the set of all maximal monochromatic components of G with respect to ω', and for each $A \in \mathcal{A}$ let c_A be the colour of A under ω'. Then, for any $d \in C$,*

$$m(G, \omega, d) \leq m(G, \omega', d) + \sum_{A \in \mathcal{A}} m(A, \omega, c_A).$$

This result means that we do not normally need to worry about the possible effect that moves played to flood a particular subgraph might have elsewhere.

Finally, we make a simple observation about the minimum number of moves required to flood a graph coloured with c colours.

Proposition 1.5. *Let G be any graph with colouring ω, where ω uses exactly c colours on G. Then*

$$m(G, \omega) \geq c - 1.$$

Moreover, if every colour appears in at least two distinct monochromatic components with respect to ω, then

$$m(G, \omega) \geq c.$$

Proof. To see that the first statement is true, note that any move can reduce the number of colours present in the graph by a most one, and so, in order to reduce then total number of colours present from c to 1, at least $c - 1$ moves are required. For the second part of the result, observe that the first move played can change only change the colour of one monochromatic component under the initial colouring, and so if every colour initially appears in at least two distinct monochromatic components then the first move cannot reduce the total number of colours present on the graph; thus a total of at least c moves will be required to reduce the number of colours in the graph to 1. □

2 Upper Bounds

In this section we derive two upper bounds on the value of $M_c(G)$, where G is an arbitrary connected graph. The first bound is more easily proved, and depends only on the number of vertices in G and the number of colours c, while the second bound depends additionally on the radius of G. In Section 3 below, we will see that these bounds are tight for particular classes of graphs.

Before giving our first bound, we prove an easy upper bound on the number of moves required to flood a graph with a specified colour.

Lemma 2.1. *Let G be any connected graph with colouring ω from colour-set C. Then, for any $d \in C$,*

$$m(G, \omega, d) \leq n - N_d(G, \omega).$$

Proof. This result follows immediately from the fact that, provided at least one vertex does not yet have colour d, it is always possible to play a move which increases the number of vertices having colour d by at least one: changing the colour of a vertex that does not already have colour d to d may additionally give some other vertices colour d, but all vertices that previously had colour d will be unchanged. $\square$

The first bound now follows easily.

Theorem 2.2. *Let G be any connected graph. Then*

$$M_c(G) \leq n - \left\lceil \frac{n}{c} \right\rceil .$$

Proof. Let ω be any colouring of G with c colours. There must then be at least one colour d such that $N_d(G, \omega) \geq \left\lceil \frac{n}{c} \right\rceil$, implying by Lemma 2.1 that $m(G, \omega, d) \leq n - \left\lceil \frac{n}{c} \right\rceil$. $\square$

We now give our second bound, which relates the maximum number of moves that may be required to flood a graph to its radius.

Theorem 2.3. *Let $G = (V, E)$ be a connected graph with radius r. Then*

$$M_c(G) \leq (c - 1)r.$$

Proof. By definition of the radius of G, there is some vertex $v \in V$ such that, for all $u \neq v \in V$, $d(u, v) \leq r$. For $1 \leq i \leq r$, let $V_i = \{u \in V : d(u, v) = i\}$, and note that $V = \{v\} \cup \bigcup_{1 \leq i \leq r} V_i$. Now fix any colouring $\omega \in \Omega(V, \{1, \ldots, c\})$. We will argue, by induction on r, that there is a sequence of moves played at v which will flood the graph. The base case, for $r = 0$, is trivial, so we will assume that $r > 0$ and that the result holds for all graphs with radius smaller than r.

Let C_1 be the set of colours, other than $\omega(v)$, that occur at vertices of V_1 under ω; note that $|C_1| \leq c - 1$. Then, cycling v through all colours in $C_!$ will create a monochromatic component containing (at least) all of $\{v\} \cup V_1$; we will denote the new colouring of G resulting from these moves by ω'. Note that the graph obtained from G by contracting monochromatic components with respect to ω' has radius at most $r-1$, so by the inductive hypothesis we see that at most $(c-1)(r-1)$ further moves are required to flood G. Hence the total number of moves required to flood G is at most $(c-1)+(c-1)(r-1) = (c-1)r$, as required. $\square$

3 Lower bounds for specific classes of graphs

In this section we prove lower bounds for some special classes of graphs, which show that the upper bounds derived in Section 2 above are tight in certain situations.

3.1 Trees

In this section we show that the upper bound on $M_c(G)$ given in Theorem 2.3 above is tight for a particular family of trees. For any $r \in \mathbb{N}$, we define $T_{c,r}$ to be the tree obtained from the star $K_{1,r(c-1)^{r+1}}$ by subdividing each edge exactly $r-1$ times (so $T_{c,r}$ is composed of $(r+1)(c-1)^{r+1}$ paths on $r+1$ vertices, all having a common first vertex). Note that the radius of $T_{c,r}$ is equal to r.

Theorem 3.1. *Let $T_{c,r}$ be as defined above. Then*

$$M_c(T_{c,r}) = (c-1)r.$$

Proof. Since the radius of $T_{c,r}$ is r, it follows from Theorem 2.3 above that $M_c(T_{c,r}) \leq (c-1)r$; thus it suffices to demonstrate a c-colouring ω of $V(T_{c,r})$ such that $m(T_{c,r}, \omega) \geq (c-1)r$.

Let v be the vertex of $T_{c,r}$ with degree r; we will set $\omega(v) = 1$. Now let $\mathcal{S}_{c,r}$ be the set of all sequences of elements from $\{1, \ldots, c\}$ of length r with the following properties:

1. the first element of the sequence is not 1, and
2. no two consecutive elements of the sequence are the same.

Note that this definition implies that $|\mathcal{S}_{c,r}| = (c-1)^r$. We will set our colouring ω to colour $(r+1)(c-1)$ of the paths in our tree with each $\sigma \in \mathcal{S}_{c,r}$: to colour a path with σ, we give the vertex adjacent to v the colour that is the first element of σ, the next vertex the colour that is the second element, and so on. Note that the conditions on elements of $\mathcal{S}_{c,r}$ ensure that this colouring ω is a proper colouring of $T_{c,r}$.

We now argue that any sequence of moves plated only at v and which floods the graph must have length at least $(c-1)r$. Let S be any sequence of moves played at v which floods G (as all moves in S are played at the same vertex, we may regard S as simply a sequence of colours). Note that there must be a colour c_1, other than 1, that is none of the first $c-2$ moves of S. We now define c_i inductively: set S_{i-1} to be the shortest initial segment of S that is a supersequence of $c_1, \ldots, c_{i-1}$, and choose c_i to be a colour, not equal to c_{i-1}, that does not appear in the first $c-2$ moves of the sequence S after S_{i-1} has been removed. Observe that $c_1, \ldots, c_r$ is an element of $\mathcal{S}_{c,r}$, so there is some path in $T_{c,r}$ whose vertices (starting from the vertex adjacent to v) are coloured, in order, $c_1, \ldots, c_r$. In order to flood this path, we must play S_r; but by construction, $|S_r| \geq (c-1)r$, so we must have $|S| \geq (c-1)r$, as claimed.

Finally, it remains to check that no sequence of fewer than $r(c-1)$ moves in which not all moves are played at v can flood the tree. Suppose that some number α of the moves are not played in monochromatic components containing v, with with $1 \leq \alpha < (c-1)r$. Note that any such move cannot change the colour of any vertex outside the path in which it is played. Thus, even if $\alpha = (c-1)r - 1$, there must still be at least one path with each colouring from $\mathcal{S}_{c,r}$ whose colouring is not changed by any of these moves that is not played at v; by the argument above, at least $(c-1)r$ moves played at v will be required to flood these remaining paths.

Hence we see that $m(T_{c,r}, \omega) \geq (c-1)r$, as required. $\square$

3.2 Paths

In this section we show that the upper bound on $M_c(G)$ given in Theorem 2.2 is tight in the case that G is a path. We prove the following result, which determines exactly the value of $M_c(P)$ for any path P.

Theorem 3.2. *Let P be a path on n vertices. Then*

$$M_c(P) = n - \left\lceil \frac{n}{c} \right\rceil .$$

This theorem follows immediately from Corollary 3.5 below, together with Theorem 2.2 above. Before proving our lower bound on the number of moves that may be required to flood a path, we define a special family of colourings of paths.

Definition. *Let $P = v_1 \ldots v_n$ be a path with edge-set $E = \{v_i v_{i+1} : 1 \leq i \leq n-1\}$, $C = \{d_0, \ldots, d_{c-1}\}$ a set of colours, and $\omega : V(P) \to C$ a proper colouring of P. The colouring ω is said to be a C-*rainbow *colouring of P if there exists a permutation $\pi : \{0, \ldots, c-1\} \to \{0, \ldots, c-1\}$ such that, for $1 \leq i \leq n$, $\omega(v_i) = d_{\pi(i \mod c)}$.*

Note that, up to relabelling of the colours, a C-rainbow coloured path must be as illustrated in Figure 1.

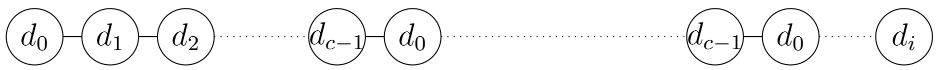

Figure 1: A C-rainbow colouring of a path

We also define a generalisation of rainbow colourings, in which the colouring my be shifted cyclically.

Definition. *Let $P = v_1 \ldots v_n$ be a path with edge-set $E = \{v_i v_{i+1} : 1 \leq i \leq n-1\}$, $C = \{d_0, \ldots, d_{c-1}\}$ a set of colours, $\omega : V(P) \to C$ a proper colouring of P, and $r \in \{0, \ldots, n-1\}$. The colouring ω is said to be an* r-shifted C-rainbow colouring *of P if there exists a permutation $\pi : \{0, \ldots, c-1\} \to \{0, \ldots, c-1\}$ such that, for $1 \leq i \leq n$,*

$$\omega(v_i) = d_{\pi((i-r) \mod c)}.$$

We say that ω is a shifted *C-rainbow colouring of P if it is an r-shifted C-rainbow colouring for some $r \in \{0, \ldots, n-1\}$.*

Note that a C-rainbow colouring is a 0-shifted C-rainbow colouring.

We now make a simple observation about the sizes of colour-classes under rainbow and shifted rainbow colourings.

Proposition 3.3. *Let ω be an r-shifted C-rainbow colouring of the path P, for some colour-set C and some $r \in \{0, \ldots, n-1\}$. Then, for any $d_1, d_2 \in C$, $|N_{d_1}(P, \omega) - N_{d_2}(P, \omega)| \leq 1$, and $\max_{d \in C} N_d(P, \omega) = \left\lceil \frac{n}{|C|} \right\rceil$.*

Proof. This is trivial in the case that ω is a C-rainbow colouring; in the case that C is an r-shifted C-rainbow colouring for $r \geq 1$, observe that deleting the edge $v_r v_{r+1}$ and adding the edge $v_n v_1$ gives a path which is C-rainbow coloured by ω, without changing the size of any colour class. $\square$

We now demonstrate that any r-shifted C rainbow colouring of a path attains the upper bound from Theorem 2.2; to do so, we first prove a lower bound on the number of moves required to flood such a coloured path in a specified colour.

Lemma 3.4. *Let P be a path on n vertices, and ω an r-shifted C-rainbow colouring of P (for some $r \in \{0, \ldots, n-1\}$), for some colour-set C with $|C| = c \geq 2$. Then, for any $d \in C$,*

$$m(P, \omega, d) \geq n - N_d(P, \omega).$$

Proof. We proceed by induction on $m(P, \omega, d)$. For the base case, suppose that $m(P, \omega, d) = 0$, which is only possible if the path is already monochromatic with colour d; thus $N_d(P, \omega) = n$ and so $n - N_d(P, \omega) = 0 \leq m(P, \omega, d)$, as required.

Now suppose that $m(P, \omega, d) > 0$ and that the result holds for any path P', colouring ω' and $d' \in C$ such that $m(P', \omega', d') < m(P, \omega, d)$. Let S be an optimal sequence of moves to flood P with colour d (starting from the initial colouring ω), and let α be the final move of S. There are now two cases to consider, depending on whether or not P is monochromatic immediately before α is played.

Suppose first that P is monochromatic in some colour $d' \neq d$ immediately before α is played. In this case we know that $m(P, \omega, d') \leq m(P, \omega, d) - 1$ and so we may apply the inductive hypothesis to see that

$$m(P, \omega, d') \geq n - N_{d'}(P, \omega).$$

Thus, by Proposition 3.3, we see that

$$\begin{aligned} m(P, \omega, d) &\geq m(P, \omega, d') + 1 \\ &\geq n - N_{d'}(P, \omega) + 1 \\ &\geq n - (N_d(P, \omega) + 1) + 1 \\ &= n - N_d(P, \omega), \end{aligned}$$

as required.

Now suppose that P is not monochromatic immediately before α is played. In this case, before the final move, there must be either two or three monochromatic segments; we denote these segments $P_1, \ldots, P_\ell$ (where $\ell \in \{2, 3\}$), and without loss of generality we may assume that P_2 does not have colour d before α is played. For each $i \in \{1, \ldots, \ell\}$, let S_i be the subsequence of $S \setminus \alpha$ consisting of moves played in a monochromatic component that intersects P_i; note that these subsequences partition $S \setminus \alpha$. Moreover, observe that S_i, played in P_i, must make P_i monochromatic; for each $i \neq 2$ the sequence S_i must flood P_i with colour d, while S_2 must flood P_2 with some colour $d' \neq d$. Thus we see that, for $i \neq 2$,

$$m(P_i, \omega, d) \leq |S_i|,$$

and also

$$m(P_2, \omega, d') \leq |S_2|.$$

Observe further that ω is a (possibly shifted) C-rainbow colouring of P_i so, as $|S_i| \leq |S \setminus \alpha| < m(P, \omega, d)$ for all i, we can then apply the inductive hypothesis to see that, for $i \neq 2$,

$$|S_i| \geq m(P_i, \omega, d) \geq |P_i| - N_d(P_i, \omega),$$

and that

$$|S_2| \geq m(P_2, \omega, d') \geq |P_2| - N_{d'}(P_2, \omega) \geq |P_2| - N_d(P_2, \omega) - 1$$

by Proposition 3.3. This then implies that

$$\begin{aligned}
m(P, \omega, d) &= |S| \\
&= 1 + \sum_{i=1}^{\ell} |S_i| \\
&= 1 + |S_2| + \sum_{i \neq 2} |S_i| \\
&\geq 1 + |P_2| - N_d(P_2, \omega) - 1 + \sum_{i \neq 2} (|P_i| - N_d(P_i, \omega)) \\
&= |P| - N_d(P, \omega),
\end{aligned}$$

as required. □

The stated lower bound on the number of moves required to flood a path with an r-shifted C-rainbow colouring now follows immediately by considering the maximum value of $N_d(P, \omega)$ taken over all $d \in C$.

Corollary 3.5. *Let P be a path on n vertices, and ω an r-shifted C-rainbow colouring of P, for some colour-set C. Then*

$$m(P, \omega) \geq n - \left\lceil \frac{n}{c} \right\rceil.$$

3.3 Cycles

In this section we build on the results concerning paths in Section 3.2 above to show that, in the worst case, the number of moves required to flood a cycle is in fact the same as that required to flood a path on the same number of vertices.

Theorem 3.6. *Let G be a cycle on n vertices. Then*

$$M_c(G) = n - \left\lceil \frac{n}{c} \right\rceil.$$

By Theorem 2.2 above, it suffices to exhibit a colouring ω of the cycle G such that $m(G, \omega) \geq n - \frac{n}{c}$. In order to do this, we extend the definition of rainbow colourings to cycles.

Definition. *Let $G = (V, E)$ be a cycle, let $V = \{v_1 \dots v_n\}$ and let $E = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1\}$. Suppose that $C = \{d_0, \dots, d_{c-1}\}$ is set of colours, and $\omega : V \to C$ is a proper colouring of V. ω is said to be a C-*rainbow *colouring of G if ω is a shifted C-rainbow colouring of the path P obtained by deleting the edge v_nv_1.*

We now demonstrate that a C-rainbow colouring of a cycle will attain the upper bound from Theorem 2.2, thus completing the proof of Theorem 3.6.

Lemma 3.7. *Let $G = (V, E)$ be a cycle on n vertices, and let ω be a C-rainbow colouring of G for some colour-set C with $|C| = c \geq 2$. Then*

$$m(G, \omega) \geq n - \left\lceil \frac{n}{c} \right\rceil.$$

Proof. Observe that every spanning tree of G is a path on n vertices; it follows from the definitions of rainbow colourings of paths and cycles that ω is an r-shifted C-rainbow colouring (for some $0 \leq r \leq n - 1$) of every spanning path of G. Thus it follows from Corollary 3.5 that, if P is any spanning path of G, $m(P, \omega) \geq n - \lceil \frac{n}{c} \rceil$. But then, by Theorem 1.1, it follows that

$$m(G, \omega) \geq n - \left\lceil \frac{n}{c} \right\rceil,$$

as required. □

4 Blow-ups of paths and cycles

In this section we determine $M_c(G)$ in the case that G is a blow-up of either a path or a cycle, provided that the length of the underlying path or cycle is sufficiently large compared with the number of colours. Specifically, we prove the following result.

Theorem 4.1. *Let G be a blow-up of a path or cycle on t vertices, and suppose that $t \geq 2c^{10}$. Then*

$$M_c(G) = t - \left\lceil \frac{t}{c} \right\rceil.$$

As in previous sections, this result is proved in two stages: we provide a lower bound on $M_c(G)$ in this setting by analysing a particular colouring of G, and then proceed to argue that the same value gives an upper bound on the number of moves required to flood G with *any* initial colouring.

Recall from the definition of a blow-up of a graph that, if G is a blow-up of a path on t vertices, the vertices of G can be partitioned into vertex-classes $V_1, \dots, V_t$ such that each class V_i is an independent set in G and uw is an edge in G if and only if $u \in V_i$ and $w \in V_j$ where $|i - j| = 1$. Similarly, if G is a blow-up of a cycle on t vertices, then the vertices of G can be partitioned into vertex classes $V_1, \dots, V_t$ such that uw is an edge in G if and only if $u \in V_i$ and $w \in V_j$ where either $|i - j| = 1$ or $\{i, j\} = \{1, n\}$.

We now define an important restricted family of colourings for graphs that are blow-ups of paths.

Definition. *Let $G = (V, E)$ be a blow-up of the path P_t, and let $C = \{d_0, \ldots, d_{c-1}\}$ be a set of colours. We say that the colouring $\omega : V \to C$ is a* path colouring *of G if there exists a function $f : \{1, \ldots, t\} \to C$ such that, for each $1 \leq i \leq t$, we have $\omega(v) = f(i)$ for every $v \in V_i$.*

Using this definition, we extend our definition of C-rainbow colourings to blow-ups of paths, to define a subfamily of path colourings.

Definition. *Let $G = (V, E)$ be a blow-up of the path P_t on t vertices. Suppose that $C = \{d_0, \ldots, d_{c-1}\}$ is a set of colours, and $\omega : V \to C$ is a path colouring of V. The colouring ω is said to be a C-*rainbow *colouring of G if the corresponding colouring of P_t is a C-rainbow colouring of the path.*

We say simply that ω is a *rainbow* colouring if it is a C-rainbow colouring for some colour-set C; the colour-set will often be clear from the context.

We now give a straightforward proof of the lower bound for blow-ups of both paths and cycles; once again, this is obtained with a rainbow colouring.

Lemma 4.2. *Let G be a blow-up of a path or cycle on t vertices. Then*

$$M_c(G) \geq t - \left\lceil \frac{t}{c} \right\rceil .$$

Proof. We define a colouring $\omega : V(G) \to \{0, \ldots, c-1\}$ by, for each $1 \leq l \leq t$, setting $\omega(v) = i \mod c$, where $v \in V_i$, for $1 \leq i \leq t$. Let G' be the graph obtained from G by adding all edges within each V_i; by Corollary 1.2 this cannot increase the number of moves required to flood the graph. Moreover, it is clear that G with colouring ω is equivalent (contracting monochromatic components) to a path or cycle on t vertices with a C-rainbow colouring. In either case, we know from Section 3 (Theorem 3.2 for paths or Theorem 3.6 for cycles) that $m(G', \omega) \geq t - \lceil \frac{t}{c} \rceil$, so we also have $m(G, \omega) \geq t - \lceil \frac{t}{c} \rceil$, as required. □

The main technical content of this section is the proof of the following upper bound for path blow-ups.

Lemma 4.3. *Let G be a blow-up of the path P_t, where $t \geq 2c^{10}$. Then*

$$M_c(G, \omega) \leq t - \left\lceil \frac{t}{c} \right\rceil .$$

The lower bound for cycle blow-ups follows easily from this result.

Corollary 4.4. *Let G be a blow-up of a cycle on t vertices, and suppose that $t \geq 2c^{10}$. Then*

$$M_c(G) \leq t - \left\lceil \frac{t}{c} \right\rceil .$$

Proof. Observe that we may delete edges from G between one pair of classes to obtain a graph G' that is a blow-up of the path P_t on t vertices. It follows immediately from Corollary 1.2 that $M_c(G) \leq M_c(G')$, and we also know from Lemma 4.3 that $M_c(G') \leq t - \lceil \frac{t}{c} \rceil$, giving the result. □

Theorem 4.1 then follows immediately from Lemma 4.2 together with Lemma 4.3 and Corollary 4.4.

We have made no attempt to optimise the dependence of t on c in the statement of Lemma 4.3, and indeed conjecture that the result is true for much smaller values of t. However, it is clear that some dependence on c is necessary, as it follows from Proposition 1.5 that if G is a blow-up of a path on c vertices in which every vertex class has size at least two and ω is a C-rainbow colouring of G (for some C with $|C| = c$) then $m(G, \omega) \geq c > c - \lceil \frac{c}{c} \rceil$. Moreover, for blow-ups of paths of length c, a C-rainbow colouring is not even the worst: if the colour class V_i contains two vertices that are assigned colours $(2i-1) \mod c$ and $(2i) \mod c$, it is straightforward to verify that $c+1$ moves are required to flood the graph, whereas c will suffice in the case of a C-rainbow colouring.

The remainder of this section is devoted to the proof of Lemma 4.3. This is done in three stages, showing first in Section 4.1 that the bound holds for rainbow colourings, extending this to all path colourings in Section 4.2, and finally generalising to arbitrary colourings in Section 4.3.

4.1 Upper bound for rainbow colourings

In this section, we prove the following result.

Lemma 4.5. *Let G be a blow-up of the path P_t, and let ω be a C-rainbow colouring of G. Then, if $t \geq c + 2$, we have*

$$m(G, \omega) \leq t - \left\lceil \frac{t}{c} \right\rceil.$$

Proof. We prove the result by induction on t. We begin by considering several base cases, which together cover the situation in which $c + 2 \leq t \leq 3c + 1$. As usual, we will denote by $V_1, \ldots, V_t$ the vertex-classes of G, and we may assume without loss of generality that V_i receives colour $i \mod c$ under ω.

For the first of these cases, suppose that $c + 2 \leq t \leq 2c$. We describe a strategy to flood G with $t - \lceil \frac{t}{c} \rceil = t - 2$ moves; this strategy is illustrated in Figures 2 and 3. First, we play $c-1$ moves at some vertex $v \in V_2$, giving this vertex colours $3, 4, \ldots, 0, 1$ in turn; this will create a monochromatic component containing v and all of $V_1 \cup V_3 \cup \cdots \cup V_{c+1}$. Note that the only vertices of $V_1 \cup \cdots \cup V_{c+1}$ that do not belong to this monochromatic component are in V_2 and have colour 2. Now we change the colour of this component to take colours $2, \ldots, t \mod c$ in turn; this will extend our monochromatic component to contain all of $V_{c+2} \cup \cdots \cup V_t$, and as the component will take colour 2 at some point in this sequence all remaining vertices of V_2 will also be flooded. Thus we have described a sequence of $c - 1 + t - (c + 1) = t - 2$ moves which floods G, as required.

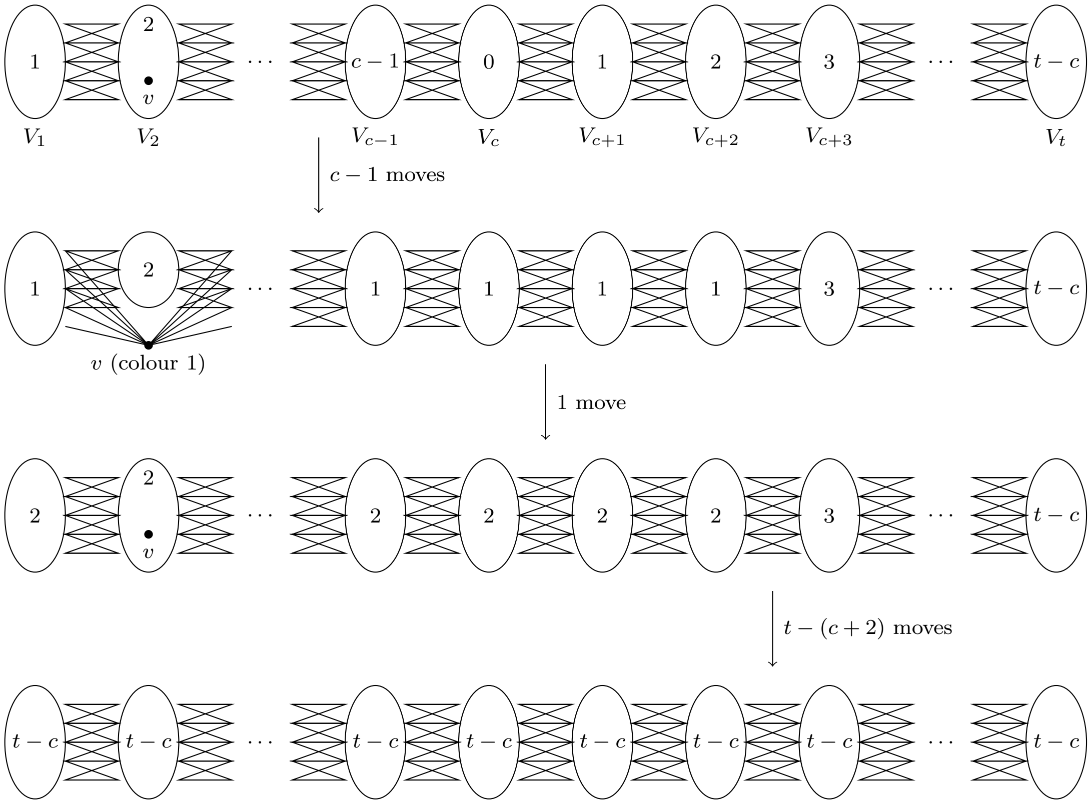

Figure 2: The first base case for Lemma 4.5: detailed version

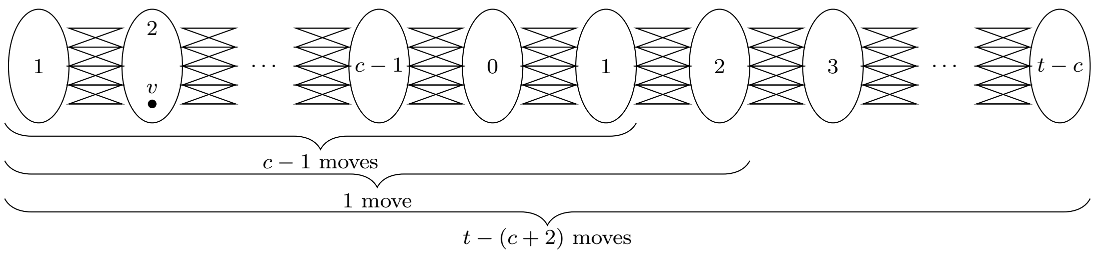

Figure 3: A reduced schematic of the first base case of Lemma 4.5

For the second base case, suppose that $2c+1 \leq t \leq 3c$. In this case we play $t - \left\lceil \frac{t}{c} \right\rceil = t-3$ moves, all at a vertex $v \in V_{c+1}$, as illustrated in Figure 4. We first change the colour of v to take colours $2, 3, \ldots, 0$ in turn; this creates a monochromatic component containing v and all of $V_c \cup V_{c+2} \cup \cdots \cup V_{2c}$. Next we give v colours $c-1, c-2, \ldots, 2$ in turn; at this point there is a monochromatic component containing all of $V_2 \cup \cdots \cup V_{2c}$ except for some vertices in V_{c+1} of colour 1. Playing one further move to give this component colour 1 therefore creates a monochromatic component containing all of $V_1 \cup \cdots \cup V_{2c+1}$. To flood the remainder of G, we give this component colours $2, \ldots, t \mod c$ in turn. This strategy allows us to flood G in a total of $c-1+c-2+1+t-(2c+1) = t-3$ moves, as required.

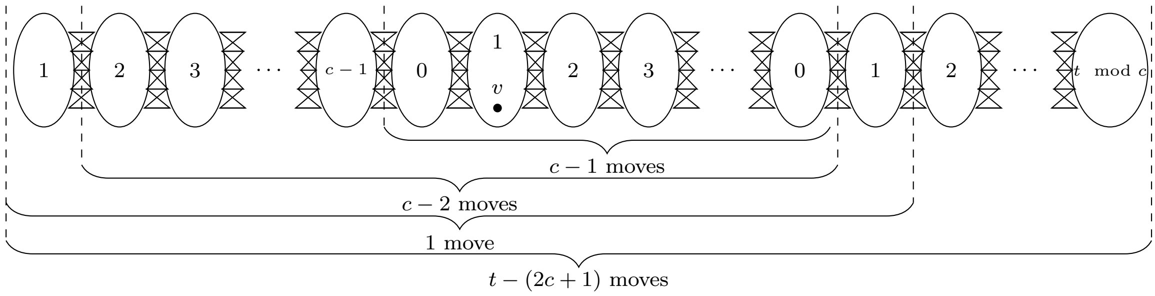

Figure 4: The second base case for Lemma 4.5

For the final base case, suppose that $t = 3c+1$. In this case we play $t - 4 = t - \left\lceil \frac{t}{c} \right\rceil$ moves, all at a vertex $v \in V_{c+2}$, as illustrated in Figure 5. We begin by giving v colours $3, \ldots, c-1, 0, 1$ in turn, which creates a monochromatic component containing all of $V_{c+1} \cup \cdots \cup V_{2c+1}$ except for vertices in V_{c+2} having colour 2. We play a further $c-1$ moves in this component, giving it colours $0, c-1, \ldots, 2$ in turn, which creates a monochromatic component containing all of $V_2 \cup \cdots \cup V_{2c+2}$. Finally, we give this component colours $3, \ldots, c-1, 0, 1$ in turn, which floods the remainder of the graph. Thus we can flood G with a total of $c-1+c-1+c-1 = 3c-3 = t-4$ moves, as required.

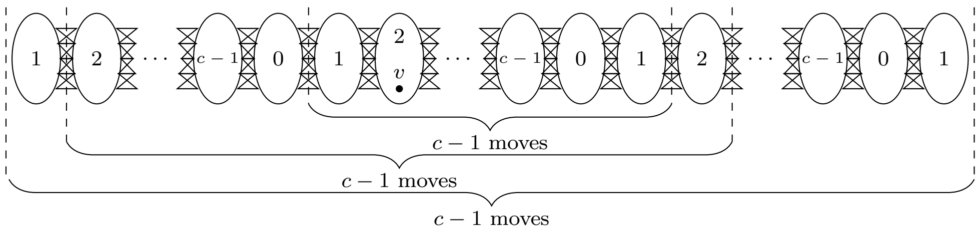

Figure 5: The third base case for Lemma 4.5

From now on, therefore, we may assume that $t \geq 3c+2$, and that the result holds for any graph that is a blow-up of P_s for $s < t$. Using a similar strategy to that described in the second base case above, we can play $2c-2$ moves which create a monochromatic component in G containing all of $V_1 \cup \cdots \cup V_{2c+1}$. To achieve this, we play $2c-1$ moves at a vertex $v \in V_{c+1}$: we give this vertex colours $2, \ldots, c-1, 0$, followed by $c-1, \ldots, 2, 1$. Note that the resulting monochromatic component ends up with colour 1, and that playing the sequence of moves described above in G will not change the colour of any vertex

outside $V_1 \cup \cdots \cup V_{2c+1}$. Thus, after playing this sequence, the resulting coloured graph is equivalent to a graph G' with colouring ω', where G' is a blow-up of the path P_{t-2c} and ω' is a C-rainbow colouring of G'. Note that $t-2c \geq c+2$, so we can apply the inductive hypothesis to see that

$$m(G', \omega') \leq t - 2c - \left\lceil \frac{t-2c}{c} \right\rceil = t - 2c + 2 - \left\lceil \frac{t}{c} \right\rceil .$$

This implies that

$$m(G, \omega) \leq 2c - 2 + t - 2c + 2 - \left\lceil \frac{t}{c} \right\rceil = t - \left\lceil \frac{t}{c} \right\rceil ,$$

as required. □

4.2 Upper bound for path colourings

Before proving that this bound is also valid for path blow-ups with any path colouring (provided that the path is sufficiently long compared with the number of colours), we need some auxiliary results. First of all, it is straightforward to verify the following characterisation of path colourings that are *not* rainbow colourings.

Proposition 4.6. *Let G be a blow-up of the path P_t, let $f : \{1, \ldots, t\} \to \{0, \ldots, c-1\}$ be any function and $C = \{d_0, \ldots, d_{c-1}\}$ a set of colours, and let ω be defined by setting $\omega(u) = d_{f(i)}$ for all $u \in V_i$ (for $1 \leq i \leq t$). If ω is not a C-rainbow colouring of G, then there exists $1 \leq i < j \leq t$ such that $j - i < c$ and $f(i) = f(j)$.*

Next we give a general bound on the number of moves required to flood a graph that is a blow-up of a path.

Proposition 4.7. *Let $G = (V, E)$ be a blow-up of the path P_t, and let $\omega : V(G) \to C$ be a colouring of G. Suppose that Q is a subpath of G containing precisely one vertex from each vertex class. Then*

$$m(G, \omega) \leq m(Q, \omega) + (c - 1),$$

and in particular

$$m(G, \omega) \leq t - \left\lceil \frac{t}{c} \right\rceil + (c - 1).$$

Proof. Note that, by Lemma 1.2, $m_Q(Q, \omega) \leq m_G(Q, \omega)$, so it is possible to play at most $m(Q, \omega)$ moves in G to create a monochromatic component A of some colour $d \in C$, where A contains all of Q and in particular contains at least one vertex from each vertex class. Thus every vertex in G either belongs to A or has a neighbour in A. We can therefore flood the remainder of G with at most $c-1$ further moves, changing the colour of A repeatedly to give it every colour in $C \setminus \{d\}$. This implies that

$$m(G, \omega) \leq m(Q, \omega) + (c - 1),$$

as required. The second part of the result then follows immediately from Theorem 3.2. □

We also need one more result relating the number of moves required to flood a path an a collection of sub-paths.

Lemma 4.8. *Let P be a path with colouring ω from colour-set C, where $|C| = c$, and let $Q_1, \ldots, Q_r$ be a collection of disjoint sub-paths of P. Then*

$$m(P, \omega) \leq t - \sum_{i=1}^{r}(|Q_i| - 1) - \left\lceil \frac{t - \sum_{i=1}^{r}(|Q_i| - 1)}{c} \right\rceil + \sum_{i=1}^{r} m(Q_i, \omega).$$

Proof. For each $1 \leq i \leq r$, fix $d_i \in C$ such that $m(Q_i, \omega) = m(Q_i, \omega, d)$. We now define a new colouring ω' of P by setting

$$\omega'(v) = \begin{cases} d_i & \text{if } v \in Q_i \\ \omega(v) & \text{otherwise.} \end{cases}$$

Observe that P with colouring ω' is equivalent to a path on at most $t - \sum_{i=1}^{r}(|Q_i| - 1)$ vertices, so by Theorem 3.2 we have

$$m(P, \omega') \leq t - \sum_{i=1}^{r}(|Q_i| - 1) - \left\lceil \frac{t - \sum_{i=1}^{r}(|Q_i| - 1)}{c} \right\rceil.$$

Let $\mathcal{A}$ be the set of maximal monochromatic components of P with respect to ω', where each $A \in \mathcal{A}$ has colour d_A under ω. Then, by Lemma 1.4, we have

$$\begin{aligned} m(P, \omega) &\leq m(P, \omega') + \sum_{A \in \mathcal{A}} m(A, \omega, d_A) \\ &\leq t - \sum_{i=1}^{r}(|Q_i| - 1) - \left\lceil \frac{t - \sum_{i=1}^{r}(|Q_i| - 1)}{c} \right\rceil + \sum_{A \in \mathcal{A}} m(A, \omega, d_A). \end{aligned}$$

So it remains to check that $\sum_{A \in \mathcal{A}} m(A, \omega, d_A) \leq \sum_{i=1}^{r} m(Q_1, \omega)$. Note that it is possible that more than one of the subpaths $Q_1, \ldots, Q_r$ belongs to the same maximal monochromatic component with respect to ω'; suppose that $A_1, \ldots, A_s$ are the elements of $\mathcal{A}$ that contain at least one subpath Q_i, and observe therefore that $s \leq r$. Moreover, it is clear that, for each $1 \leq i \leq s$,

$$m(A_i, \omega, d_{A_i}) \leq \sum_{Q_j \subseteq A_i} m(Q_j, \omega, d_i) = \sum_{Q_j \subseteq A_i} m(Q_j, \omega).$$

Observe also that, for $A \in \mathcal{A}$ with $A \notin \{A_1, \ldots, A_s\}$, A is also a monochromatic component of P with respect to ω, so we have $m(A, \omega, d_A) = 0$. Hence, as no subpath Q_j belongs to more than one monochromatic component A_i, we have

$$\sum_{A \in \mathcal{A}} m(A, \omega, d_A) \leq \sum_{i=1}^{s} \sum_{Q_j \subseteq A_i} m(Q_i, \omega) = \sum_{i=1}^{r} m(Q_i, \omega),$$

completing the proof. $\square$

We now use these auxiliary results to extend our upper bound to cover all initial colourings that are path colourings. The key idea of the proof is to define a quantity that captures in a sense how far away the initial colouring is from a rainbow colouring. If the initial colouring is sufficiently different from a rainbow colouring, we can argue that we must be able to create a monochromatic end-to-end path significantly more quickly than in the rainbow case, meaning that we are then able to flood any remaining vertices greedily. In the event that the colouring does not differ so much from a rainbow colouring, we demonstrate how we may perform a sequence of flooding moves that is not too long and which results in a rainbow-coloured path blow-up, allowing us to apply the previous result.

Lemma 4.9. *Let G be a blow-up of the path P_t, let $c \geq 3$ and let $f : \{1, \ldots, t\} \to \{1, \ldots, c\}$ be any function, and let ω be defined by setting $\omega(u) = f(i)$ for all $u \in V_i$ (for $1 \leq i \leq t$). Then, if $t \geq 2c^2(c-1)^3$,*

$$m(G, \omega) \leq t - \left\lceil \frac{t}{c} \right\rceil .$$

Proof. Without loss of generality, we may assume that, for every $1 \leq x < t$, $f(x) \neq f(x+1)$, as otherwise we could contract all vertices of $V_x \cup V_{x+1}$ to a single vertex, obtaining an equivalent coloured graph which is a blow-up of a path on $t-1$ vertices. Now observe that, for any path colouring ω of G, the graph G can be decomposed into subgraphs $G_1, \ldots, G_r$, where each G_i is a blow-up of the path P_{t_i} and $\sum_{i=1}^{r} t_i = t$, in such a way that ω is a C-rainbow colouring of G_i for each $1 \leq i \leq r$; it is clear that such a decomposition must exist since setting $G_i = G[V_i]$ for $1 \leq i \leq t$ will do.

A more meaningful decomposition with the required properties can be constructed greedily, as illustrated in Figure 6: we choose t_1 to be the largest integer such that ω is a C-rainbow colouring of $G[V_1 \cup \cdots \cup V_{t_1}]$, and given $t_1, \ldots, t_i$ we set $s_{i+1} = 1 + \sum_{j=1}^{i} t_j$ and choose t_{i+1} to be the largest integer such that ω is a rainbow colouring of $G[V_{s_{i+1}} \cup \cdots \cup V_{s_{i+1}+t_{i+1}-1}]$. We call the decomposition constructed in this way the *greedy rainbow decomposition of* (G, ω), and denote by $\operatorname{grd}(G, \omega)$ the collection of subgraphs in this decomposition. Then $\operatorname{grd}(G, \omega) = \{G_1, \ldots, G_r\}$ (for some $r \geq 1$), where $G_i = G[V_{s_i} \cup V_{s_{i+1}} \cup \cdots \cup V_{s_{i+1}-1}]$, and the greedy construction guarantees that, for each $1 \leq i \leq r = |\operatorname{grd}(G, \omega)|$, there exists x_i with $\max\{s_i, s_{i+1} - c + 1\} \leq x_i \leq s_{i+1} - 1$ such that $f(x_i) = f(s_{i+1})$.

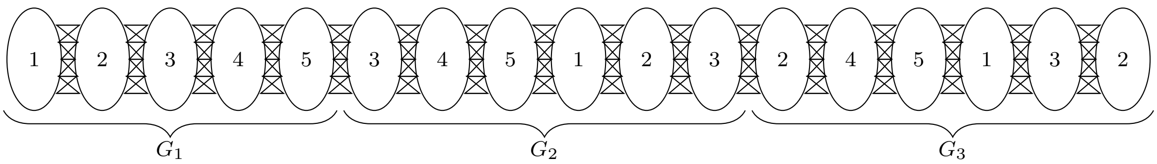

Figure 6: An example of a greedy rainbow decomposition

Suppose first that $|\operatorname{grd}(G, \omega)| > 2c(c-1)$. We will argue that in this case we can flood G in at most $m(G, \omega)$ moves by first creating a monochromatic end-to-end path and then cycling through any remaining colours.

Fix Q to be any path that contains precisely one vertex from each vertex class $V_1, \dots, V_t$. Now, for $1 \le i \le r-1$, set Q_i to be the segment of Q induced by $Q \cap (V_{x_i} \cup \dots \cup V_{s_{i+1}})$. Observe that for each Q_i, by definition of x_i, we have $|Q_i| \le c$, and $m(Q_i, \omega, f(x_i)) \le |Q_i| - 2$, by Lemma 2.1. We are not quite able to apply Lemma 4.8, as for any i it is possible that Q_i and Q_{i+1} intersect in one vertex; however, it is clear that $Q_i \cap Q_{i+2} = \emptyset$ for any i, so it is certainly the case that $\{Q_{2i} : 1 \le i \le \left\lfloor \frac{|\operatorname{grd}(G,\omega)|}{2} \right\rfloor\}$ is a collection of disjoint sub-paths of Q. Setting $r = |\operatorname{grd}(G,\omega)|$, Lemma 4.8 now tells us that

$$
\begin{aligned}
m(Q,\omega) &\le t - \sum_{i=1}^{\lfloor \frac{r}{2} \rfloor} (|Q_{2i}| - 1) - \left\lceil \frac{t - \sum_{i=1}^{\lfloor \frac{r}{2} \rfloor} (|Q_{2i}| - 1)}{c} \right\rceil + \sum_{i=1}^{\lfloor \frac{r}{2} \rfloor} m(Q_{2i}, \omega) \\
&\le t - \left\lceil \frac{t}{c} \right\rceil - (c-1),
\end{aligned}
$$

since $|Q_i| \le c$, $m(Q_{2i}, \omega) \le |Q_i| - 2$, and $r = |\operatorname{grd}(G,\omega)| \ge 2c(c-1)$. Proposition 4.7 then gives $m(G,\omega) \le t - \lceil \frac{t}{c} \rceil$, as required.

So we may assume from now on that $|\operatorname{grd}(G,\omega)| \le 2c(c-1)$. In this case it suffices to prove the following claim, by our initial assumption that $t \ge 2c^2(c-1)^3$.

Claim. *If $t \ge c(c-1)^2 |\operatorname{grd}(G,\omega)|$ then*

$$
m(G,\omega) \le t - \left\lceil \frac{t}{c} \right\rceil .
$$

The base case, for $|\operatorname{grd}(G,\omega)| = 1$, follows from Lemma 4.5 (since in this case ω must be a rainbow colouring of G), so we may assume from now on that $|\operatorname{grd}(G,\omega)| \ge 2$ and that the claim holds for any such graph $\widetilde{G}$ with colouring $\widetilde{\omega}$ such that $|\operatorname{grd}(\widetilde{G},\widetilde{\omega})| < |\operatorname{grd}(G,\omega)|$.

By our assumption that $t \ge c(c-1)^2 |\operatorname{grd}(G,\omega)|$, there must be some $1 \le i \le |\operatorname{grd}(G,\omega)|$ such that $t_i \ge c(c-1)^2 > 2c$. Note that we may assume without loss of generality that $i \ne |\operatorname{grd}(G,\omega)|$: if the only such subgraph in the decomposition is $G_{|\operatorname{grd}(G,\omega)|}$ then we can reverse the ordering of the vertex classes so that the longest subgraph in the decomposition is instead G_1; the subgraphs of the decomposition may not be the same as before, but our longest section can only increase in length in this new setting. We will now consider the subgraph $H_0 = G[V(G_i) \cup V(G_{i+1})]$, where ω_0 is the restriction of ω to $V(H_0)$.

We will describe how to play a sequence of moves in H_0 which results in a rainbow colouring of this subgraph; if this does not decrease the length of the underlying path too much (when monochromatic components are contracted) we then invoke the inductive hypothesis, and otherwise we can complete the proof directly. Specifically, we describe how to obtain a sequence of graphs $(H_j)_{j=1}^{s}$ (for some $s \ge 1$), with corresponding colourings $(\omega_j)_{j=1}^{s}$, with three key properties.

We denote by $U_1^{(0)}, \dots, U_{\ell_0}^{(0)}$ the vertex classes of H_0, where $\ell_0 = t_i + t_{i+1}$, and let $f_0 : \{1, \dots, \ell_0\} \to C$ be the function such that, for every $1 \le z \le \ell_0$, $\omega_0(u) = f_0(z)$ for each $u \in U_z^{(0)}$. For each j, the coloured graph (H_j, ω_j) then has the following properties:

1. H_j is a blow-up of a path P_{ℓ_j}, where $\ell_j \geq \ell_0 - (j+1)(c-1) - 1$, and ω_j is a proper path colouring of H_j,

2. There exists $x_j \in \{1, \ldots, \ell_j\}$ such that, if the vertex classes of H_j are $U_1^{(j)}, \ldots, U_{\ell_j}^{(j)}$, then $|U_{x_j}^{(j)}| = 1$ and ω_j is a C-rainbow colouring of both $H_j[U_1^{(j)} \cup \cdots \cup U_{x_j}^{(j)}]$ and $H_j[U_{x_j}^{(j)} \cup \cdots \cup U_{\ell_j}^{(j)}]$ (as illustrated in Figure 7), and

3. If the colouring ω'_j of $V(H_0)$ is defined by

$$\omega'_j(v) = \begin{cases} \omega_0(v) & \text{if } v \in U_z^{(0)} \text{ and either } z < x_j \text{ or } z > \ell_0 - (\ell_j - x_j) \\ \omega_j(x_j) & \text{otherwise,} \end{cases}$$

then H_0 with colouring ω'_j is equivalent to H_j with colouring ω_j; moreover, if $\mathcal{A}_j$ is the set of maximal monochromatic components of H_0 with respect to ω'_j, where each $A \in \mathcal{A}_j$ has colour d_A^j under ω'_j, then $\sum_{A \in \mathcal{A}_j} m(A, \omega_0, d_A^j) \leq \ell_0 - \ell_j - j - 1$.

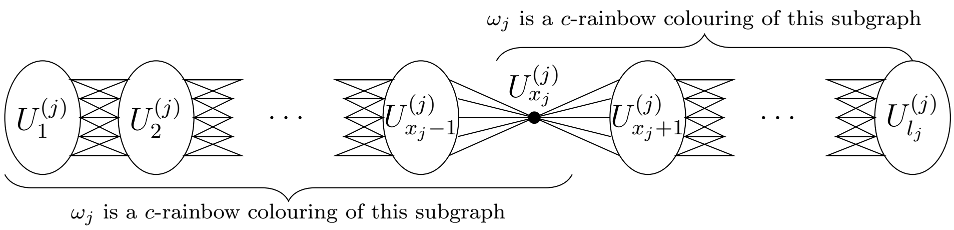

Figure 7: The structure of H_j

Informally, we repeatedly perform moves to create a single monochromatic component at the boundary of the two rainbow-coloured segments (whose colour is consistent with both rainbow colourings), taking care to make sure that we do not play too many moves or shorten the path too much at any stage. The key idea is to exploit the fact that some colour must occur more frequently, as we traverse the subgraph from left to right, than would happen under a rainbow colouring.

We describe in detail how to obtain the first pair (H_1, ω); the method for constructing further pairs is very similar but somewhat simpler. Throughout, the only assumption we require, in addition to the fact that (H_j, ω_j) has the three stated properties, is that ω_j is not a rainbow colouring of H_j.

By construction of $\mathrm{grd}(G, \omega)$, we know that there is some $y \in \{t_i - c + 2, \ldots, t_i\}$ such that $f_0(y) = f_0(t_i + 1)$: if not, then we would have chosen G_i to include at least one more vertex class. In fact, by our assumption that no two consecutive vertex classes receive the same colour under ω, we know that $y \in \{t_i - c + 2, \ldots, t_i - 1\}$. Since ω_0 is a C-rainbow colouring of G_i, and $t_i > 2c$, we also know that $f_0(y - c) = f_0(y) = f_0(t_i + 1)$. Now set $F_0 = H_0[U_{y-c}^{(0)} \cup \cdots \cup U_{t_i+1}^{(0)}]$. We now describe a sequence of moves to flood F_0 with colour $f(y)$ in at most $t_i + c - y - 2$ moves, all played at some vertex $v_0 \in U_y^{(0)}$; this sequence of moves is illustrated in Figure 8. We begin by giving v_0 colours $f_0(y-1), \ldots, f_0(y-c+1) = f(y+1)$ in turn; this will create a monochromatic component containing v_0 and all of $U_{y-c+1}^{(0)} \cup \cdots \cup U_{y-1}^{(0)} \cup U_{y+1}^{(0)}$, and so that the only

vertices of $U^{(0)}_{y-c+1} \cup \cdots \cup U^{(0)}_{y+1}$ not linked to this component have colour $f_0(y)$. We then give this component colours $f_0(y+2), \ldots, f_0(t_i+1) = f_0(y) = f_0(y-c)$ in turn, which will clearly flood all remaining vertices in F_0. The total number of moves played is therefore $c - 1 + t_i - y = t_i + c - y - 1$, implying that $m(F_0, \omega_0, f_0(y)) \leq t_i + c - y - 1$.

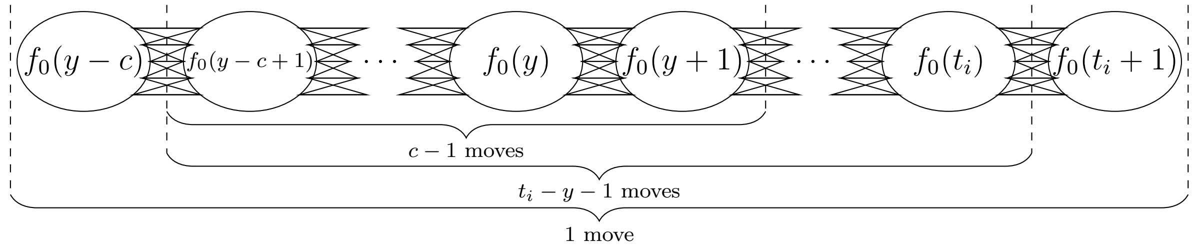

Figure 8: Flooding the subgraph F_0

Now define H_1 to be the graph obtained from H_0 by contracting all vertices of F_0 to a single vertex w_1, and ω_1 to be the colouring of H_1 that agrees with ω_0 on all vertices of H_1 except w_1, and gives w_1 colour $f_0(y)$. We claim that H_1 with colouring ω_1 has the three properties listed above. It is clear that H_1 is a blow-up of a path on

$$\ell_1 = \ell_0 - (t_i + 1 - y + c) \geq \ell_0 - 2c + 1 = \ell_0 - (1+1)(c-1) - 1$$

vertices and that ω_1 is a proper path colouring of H_1, as required to satisfy the first condition. For the second condition, set $x_1 = y - c$, and note that $U^{(1)}_{x_1} = \{w_1\}$. Further define $f_1 : \{1, \ldots, \ell_1\} \to C$ to be the function so that $\omega_1(u) = f_1(z)$ for every $u \in U^{(1)}_z$, for $1 \leq z \leq \ell_1$. Observe that $f_1(z) = f_0(z)$ for $1 \leq z \leq y - c$, so it follows from the fact that ω_0 is a C-rainbow colouring of $H_0[U^{(0)}_1 \cup \cdots \cup U^{(0)}_{t_i}] \supset H_0[U^{(0)}_1 \cup \cdots \cup U^{(0)}_{y-c}]$ that ω_1 is a C-rainbow colouring of $H_1[U^{(1)}_1 \cup \cdots \cup U^{(1)}_{x_1}]$. Moreover, for $y - c \leq z \leq \ell_1$, we see that $f_1(z) = f_0(z + \ell_0 - \ell_1)$, so the fact that ω_0 is a C-rainbow colouring of $H_0[U^{(0)}_{t_i+1} \cup \cdots \cup U^{(0)}_{\ell_0}]$ implies that ω_1 is a C-rainbow colouring of

$$H_1[U^{(1)}_{t_i+1-(\ell_0-\ell_1)} \cup \cdots \cup U^{(0)}_{\ell_0-(\ell_0-\ell_1)}] = H_1[U^{(1)}_{x_1} \cup \cdots \cup U^{(0)}_{\ell_1}].$$

Thus the second condition holds. Finally, for the third condition, it is clear from the definition of ω_1 that H_1 with colouring ω_1 is equivalent to the graph H_0 with colouring ω'_1 (with ω'_1 defined with respect to ω_1 as in the statement of the third condition); note also that the only maximal monochromatic component of H_0 with respect to ω'_1 that is not also a maximal monochromatic component with respect to ω_0 is F_0 (and it is straightforward to verify that F_0 is indeed a maximal monochromatic component of H_0 with respect to ω'_1). Thus, if $\mathcal{A}_1$ denotes the set of maximal monochromatic components of H_0 with respect to ω'_1 and each $A \in \mathcal{A}_1$ has colour d_A under ω'_1, we see that

$$\begin{aligned}\sum_{A \in \mathcal{A}_1} m(A, \omega_0, d_A) &= m(F_0, \omega_0, \omega_1(x_j)) \\ &= m(F_0, \omega_0, f_0(y)) \\ &\leq t_i + c - y - 1 \\ &= (t_i + 1 - y + c) - 2 \\ &= \ell_0 - \ell_1 - 2,\end{aligned}$$

as required to satisfy the third condition. This completes the definition of H_1 and ω_1.

Suppose we keep following this general procedure to obtain pairs (H_j, ω_j) for $1 \leq j \leq s$, where s is as large as possible; by maximality of s, we may assume that ω_s is a rainbow colouring of H_s, as otherwise we could continue. (Note that if our colouring is not a rainbow colouring, this imposes a minimum condition on the length of the path, so we do not need to consider separately the possibility of our path becoming too short to apply the procedure.) We define ω' to be the colouring of G which agrees with ω on all vertices that do not belong to H_0, and with ω'_s on all vertices of H_0. Note that the maximal monochromatic components of G with respect to ω' that are *not* also maximal monochromatic components with respect to ω are precisely the maximal monochromatic components of H_0 with respect to $\omega' = \omega'_s$ (and recall also that ω_0 is the restriction of ω to H_0). Thus we can apply Lemma 1.4 to see that

$$\begin{aligned} m(G,\omega) &\leq m(G,\omega') + \sum_{A \in \mathcal{A}_s} m(A, \omega_0, d^s_A) \\ &\leq m(G,\omega') + \ell_0 - \ell_s - s - 1 \end{aligned} \tag{4.1}$$

by the third condition on (H_s, ω_s). There are now two cases to consider, depending on the value of s.

First, suppose that $s \geq c(c-1)$. In this case we argue that we can continue by creating a monochromatic end-to-end path and then cycling through any remaining colours. Let Q be a path in G which contains precisely one vertex from each class. Note that there will be a segment of $\ell_0 - \ell_s + 1$ consecutive vertices on Q which have the same colour under ω' so, under this colouring, Q is equivalent to a path of length $Q - \ell_0 + \ell_s$; Proposition 4.7 therefore implies that

$$m(G,\omega') \leq t - \ell_0 + \ell_s - \left\lceil \frac{t - \ell_0 + \ell_s}{c} \right\rceil + c - 1.$$

Substituting this in (4.1) and using both the fact that $\ell_s \geq \ell_0 - (j+1)(c-1) - 1$ (the first condition on (H_s, ω_s)) and the assumption that $s \geq c(c-1)$, this gives

$$m(G,\omega) \leq t - \left\lceil \frac{t}{c} \right\rceil,$$

as required.

Now suppose instead that $s < c(c-1)$; in this case we invoke the inductive hypothesis. Since ω_s is a C-rainbow colouring of H_0, it is clear that $|\operatorname{grd}(G,\omega')| < |\operatorname{grd}(G,\omega)|$. Moreover, G with colouring ω' is equivalent to a graph $\widetilde{G}$ with colouring $\widetilde{\omega}$, where $\widetilde{G}$ is a blow-up of a path r vertices with

$$\begin{aligned} r &= t - (\ell_0 - \ell_s) \\ &> c(c-1)^2 |\operatorname{grd}(G,\omega')| \end{aligned}$$

(making use of our assumption on the value of t and the fact that $|\operatorname{grd}(G,\omega')| \leq |\operatorname{grd}(G,\omega)| - 1$). Thus we can apply the inductive hypothesis to see that the claim holds for G with colouring ω', implying that

$$m(G,\omega') \leq t - (\ell_0 - \ell_s) - \left\lceil \frac{t - (\ell_0 - \ell_s)}{c} \right\rceil.$$

Substituting into (4.1), then gives the required result, completing the proof of the claim, and hence proving the result. □

4.3 Arbitrary colourings

In this final section, we extend the results of the previous sections to prove the upper bound for all initial colourings. The structure of this proof is in some ways similar to the previous result: we define a notion of the distance of a colouring from a path colouring, and then consider two cases. If the colouring differs sufficiently from a path colouring, we can quickly create an end-to-end path and flood any remaining vertices greedily, whereas if our initial colouring is sufficiently close to a path colouring we demonstrate how to play a sequence of moves that results in a path-coloured graph, allowing us to apply the previous result.

Proof of Lemma 4.3. let ω be any colouring of G from colour-set $C = \{1, \ldots, c\}$. We begin by setting

$$\theta(G, \omega) = |\{i : 1 \leq i \leq t \text{ and } \omega \text{ is not constant on } V_i\}|.$$

Suppose first that $\theta(G, \omega) \geq c(c-1)$, and set

$$n_j = |\{i : 1 \leq i \leq t \text{ and } \exists u \in V_i \text{ with } \omega(u) = j\}.$$

Since there are $\theta(G, \omega)$ vertex classes that each contain vertices of at least two distinct colours, we see that

$$\sum_{j=1}^{c} n_j \geq t + \theta(G, \omega) \geq t + c(c-1).$$

Thus there exists some $j \in \{1, \ldots, c\}$ such that $n_j \geq \lceil \frac{t}{c} \rceil + (c-1)$. Observe therefore that there exists a path Q containing precisely one vertex from each vertex class $V_1, \ldots, V_t$ and so that at least $\lceil \frac{t}{c} \rceil + (c-1)$ vertices on Q have colour j under ω. It then follows from Lemma 2.1 that $m(Q, \omega) \leq t - \lceil \frac{t}{c} \rceil - (c-1)$, so Proposition 4.7 gives

$$m(G, \omega) \leq t - \left\lceil \frac{t}{c} \right\rceil - (c-1) + (c-1) = t - \left\lceil \frac{t}{c} \right\rceil,$$

as required.

Thus from now on we will assume that $\theta(G, \omega) < c(c-1)$. In this case it clearly suffices to prove the following claim, since $t \geq 2c^{10} > 2c^8(\theta(G, \omega) + 1)$.

Claim. *Suppose that $t > 2c^8(\theta(G, \omega) + 1)$. Then*

$$m(G, \omega) \leq t - \left\lceil \frac{t}{c} \right\rceil.$$

We prove the claim by induction on $\theta(G, \omega)$. In the base case, for $\theta(G, \omega) = 0$, we know that ω must in fact be a path-colouring of G and so the result follows immediately from Lemma 4.9. Thus we may assume that $\theta(G, \omega) \geq 1$ and that the result holds for any graph G' with colouring ω' such that $\theta(G', \omega') < \theta(G, \omega)$.

Since $t \geq 2c^8(\theta(G,\omega)+1)$, there exists some vertex class V_i such that ω is not constant on V_i, but for $1 \leq j \leq 2c^8$ we either have ω constant on every V_{i+j}, or else ω is constant on every V_{i-j}; reversing the order of the vertex classes if necessary, we may assume without loss of generality that we have ω constant on every V_{i+j} for $1 \leq j \leq 2c^8$.

The first step in our strategy to flood G is to perform a series of moves in the $(2c^5 + 1)(c^2(c-1)+1)$ classes to the right of V_i, resulting in a colouring of these vertices that makes the subgraph they induce (after contracting monochromatic components) equivalent to a path.

We split this subgraph into $c^2(c-1)+1$ consecutive blocks, $B_1, \ldots, B_{c^2(c-1)+1}$, where each block consists of the subgraph induced by $2c^5+1$ consecutive vertex classes. Restricted to any block B_ℓ, ω is a path colouring, so it follows from Lemma 4.3 that B_ℓ can be made monochromatic in some colour d_ℓ with at most

$$2c^5 + 1 - \left\lceil \frac{2c^5+1}{c} \right\lceil = 2c^5 - c^4$$

moves. Let ϖ be the colouring of $V(G)$ that assigns d_ℓ to every vertex of $B_{|ell}$ (for $1 \leq \ell \leq c^2(c-1)+1$), and agrees with ω elsewhere. Then Lemma 1.4 tells us that

$$m(G,\omega) \leq m(G,\varpi) + (c^2(c-1)+1)(2c^5 - 2c^4). \tag{4.2}$$

We now consider the graph G', obtained by contracting monochromatic components of G with respect to ϖ, and the corresponding colouring ω'. Note that G' is a blow-up of a path on $t' = t - 2c^5(c^2(c-1)+1)$ vertices. In the remainder of the proof we will argue that in fact $m(G',\omega') \leq t' - \lceil \frac{t'}{c} \rceil$; it is straightforward to check that substituting this bound on $m(G,\varpi)$ in (4.2) gives the required result.

We now prove this bound on $m(G',\omega')$. Recall from the construction of G' that, if the vertex classes of G' are $U_1, \ldots, U_{t'}$, we have $|U_{i+\ell}| = 1$ for $1 \leq \ell \leq c^2(c-1+1)$. We may also assume without loss of generality that the colours assigned to vertices of U_i by ω' are $\{1, \ldots, r\}$ for some $r \geq 2$.

If every colour in $\{1, \ldots, r\}$ is assigned to one of the first c vertex classes to the right of U_i by ω', then we can create a monochromatic component containing all of U_i and the c vertex classes to its right using at most $c-1$ moves: we give the unique vertex in U_{i+1} the colours of $U_{i+2}, U_{i+3}, \ldots, U_{i+c}$ in turn. Otherwise, there must be some $c \in \{1, \ldots, r\}$ that is not assigned to any of the first c vertex-classes to the right of U_i by ω'. Then, by Lemma 2.1, we can perform an "efficient flooding sequence" in which we flood this subpath on c vertices with at most $c-2$ moves (as some colour must be repeated), thus reducing the length of an end-to-end path by $c-1$. We continue in this way until either we have performed an efficient flooding operation $c(C-1)$ times, or else we are able, by the method described above, to flood U_i and the c vertex classes immediately to the right with $c-1$ moves.

In the latter case, we have played $a(c-2)+(c-1)$ moves, for some $a < c(c-1)$, to create a coloured graph equivalent to a graph $\widetilde{G}$ with colouring $\widetilde{\omega}$, where $\widetilde{\omega}$ is a blow-up of a path on $t'' = t' - (a(C-1)+c)$ vertices and $\theta(\widetilde{G},\widetilde{\omega}) < \theta(G,\omega)$.[1] It is straightforward

[1]In fact the moves we have played might have flooded a larger component than is described here; but by Lemma 1.4 this can only help us.

to verify that $t'' \geq 2c^8(\theta(\widetilde{G}, \widetilde{\omega}) + 1)$ and so it follows from the inductive hypothesis that

$$m(G', \overline{\omega}) \leq t'' - \left\lceil \frac{t''}{c} \right\rceil = t' - (a(c-1)+c) - \left\lceil \frac{t' - (a(c-1)+c)}{c} \right\rceil;$$

the required bound on $m(G', \omega')$ then follows easily from the fact that $m(G', \omega') \leq a(c-2) + (c-1) + m(\widetilde{G}, \widetilde{\omega})$.

It remains to consider the case that we terminate after performing $c(c-1)$ efficient flooding sequences. In this case, we have (after contracting monochromatic components) reduced the length of an end-to-end path by $c(C-1)^2$, with at most $c(c-1)(c-2)$ moves. Thus, by Proposition 4.7 we can flood the resulting graph with a further

$$t' - c(c-1)^2 - \left\lceil \frac{t' - c(c-1)^2}{c} \right\rceil = t' - \left\lceil \frac{t'}{c} \right\rceil - c(c-1)(c-2)$$

moves, from which the bound on $m(G', \omega')$ follows immediately, completing the proof. □

5 Conclusions and Open Problems

We have determined two general upper bounds on the maximum number of moves, taken over all possible colourings, that may be required to flood a given graph G; the second of these gives a relationship between the radius of G and the maximum number of moves that may be required to flood G. A more detailed analysis of the game played on special classes of graphs showed that our first general bound is tight in the case that the underlying graph is a path or a cycle, and that the second is tight for a particular family of trees. Finally, we determined exactly the maximum number of moves that may be required to flood G when G is a blow-up of a path or a cycle, provided that the length of the path or cycle is sufficiently large compared with the number of colours; perhaps surprisingly, these results demonstrated that in the worst case the number of moves required to flood a blow-up of a path or cycle on t vertices is in fact exactly the same as the number of moves required to flood just a path or cycle respectively on t vertices.

Our results provide a partial answer to a question raised by Meeks and Scott in [13], that of determining the maximum number of moves, taken over all possible colourings, that may be required to flood a given graph G. This general question remains open, and we mention here two specific sub-questions. First of all, given the emphasis on $k \times n$ grids in the existing algorithmic analysis of flood-filling games, a natural question is to ask what is the exact value of $M_c(G)$ in the case that G is a $k \times n$ grid? Using Lemma 2.1 and Lemma 3.4, we only have $n - \left\lceil \frac{n}{c} \right\rceil$ and $n - \left\lceil \frac{n}{c} \right\rceil + (c-1)\left\lceil \frac{k-1}{2} \right\rceil$ as lower and upper bounds, respectively. Secondly, considering general graphs, we know that the bound given in Theorem 2.3 is tight for a particular family of trees (and indeed it follows from results in [14] that a bound that is tight for some graph must in fact be tight for some tree); can better upper bounds for other classes of graphs be obtained if additional restrictions added that exclude trees, for example a condition on the density or the connectivity of the graph?

The parameterised complexity of determining whether a given coloured graph can be flooded with a specified number of moves has been studied with a wide range of different parameterisations (as in, for example, [7]), but this investigation of extremal properties

gives rise to a new natural parameterised problem: given a graph G, for which we know the value of $M_c(G)$, and a colouring ω of G, what is the (parameterised) complexity of determining whether G with initial colouring ω can be flooded in at most $M_c(G) - k$ moves, where k is the parameter? It can easily be deduced from the hardness proof for $2 \times n$ boards in [13] that determining the minimum number of moves required to flood a blow-up of a path is NP-hard, so it is already meaningful to consider this parameterised problem for graphs drawn from the classes considered here.

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